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# SUSTAINABLE DEVELOPMENT THROUGH ITERATIVE SCHEME: A COMPUTATIONAL PERSPECTIVE

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**ABSTRACT:** Contrary to the notion that mathematics and sustainable development are unrelated, mathematical tools, particularly numerical iterative procedures, play a vital role in solving complex nonlinear models that arise from various fields, including science, environment, economics, and engineering. These models are crucial for quantitatively assessing sustainability, but some cannot be solved analytically. Therefore, efficient iterative procedures are essential, as they significantly impact time and cost, two critical factors in achieving sustainable development. Minimizing computational cost is a fundamental principle in developing new iterative schemes. Since derivative evaluations increase implementation costs and can be challenging for complex functions, this study presents three classes of derivative-free iterative structures with fourth, seventh, and eighth-order convergence. Using custom-designed computer programs in the MATHEMATICA software environment, we verified the convergence of these schemes. Furthermore, we applied these schemes to recently modeled physical phenomena and compared them to existing contemporary schemes, demonstrating their effectiveness as tools for sustainable economic development.

## 1.0 INTRODUCTION

Sustainable development practice involves satisfying the needs of the present without jeopardizing the ability of future generations to meet their own needs (Soumen, 2018). Mathematical modeling is a valuable tool for understanding, predicting, and controlling development processes to achieve sustainable development.

Consider the environmental management mathematical model proposed by Ehrlich and Holdren (1971), which describes the negative impact of society on its environment as

$$I = P \times A \times T, \quad (1)$$

where P, A, and T represent population, affluence, and technology, respectively. If P(w), A(w), and T(w) are expressed as functions of time (w), the right-hand side of equation (1) becomes a nonlinear equation in terms of time (w). A problem arises in determining the time (w) at which the impact of societal activities will be negligible, ensuring that future generations can enjoy quality lives. In this case, the equation in (1) can be generally expressed as

$$\Gamma(w) = 0. \quad (2)$$

Locating the solution θ of nonlinear (NL) equations, such as equation (2), is a fundamental and daunting task in computational sciences. For NL polynomial equations of degree up to four, certain analytical methods exist for locating their solutions. However, determining the solution of a NL polynomial equation of degree more than four or NL equations involving transcendental, trigonometric, logarithmic, or exponential functions is a significant challenge. To address this challenge, Iterative Structures (IS) have been developed for locating the solutions of NL equations. The IS technique involves a recurrent formula used to approximate the solution θ of the target NL equation, yielding an improved estimate of the solution θ with each iteration until the exact value or desired solution is achieved.

Fundamentally, IS for solving NL equations can be classified into two categories based on function evaluation. The first category requires the evaluation of functions $\Gamma(\cdot)$ and their derivatives $\Gamma'(\cdot)$ (see Ogbereyivwe and Ojo-Orobosa, 2021a, 2021b; Ogbereyivwe and Izevbizua, 2022; Padilla et al., 2022, and references therein). The second category comprises IS that are derivative-free (Soleymani, 2010; Sivakumar et al., 2021; Bahgat, 2021; Zafar, 2016; Panday and Jaiswal, 2017). Because evaluating function derivatives incurs additional computational costs and obtaining derivatives for some functions can be complex, IS without derivatives have been developed.

For instance, by annihilating the first derivative $\Gamma'(w)$ in the classical Newton iterative structure (NIS):

$$w_{n+1} = w_n - \frac{\Gamma(w_n)}{\Gamma'(w_n)}, \quad n = 0,1,2,\ldots \quad (3)$$

using the divided difference technique, the classical derivative-free Steffensen iterative structure (SIS) (Steffensen, 1933) is obtained:

$$w_{n+1} = w_n - \frac{(\Gamma(w_n))^2}{\Gamma(w_n+\Gamma(w_n))-\Gamma(w_n)}. \quad (4)$$

The SIS has a convergence order of two and, since it requires the evaluation of two functions per iteration, it retains the optimality property of the NIS in the view of the Kung-Traub conjecture (Kung and Traub, 1974). The conjecture states that an IS consuming n-function evaluations per iteration is optimal when it achieves a convergence order of $2^{n-1}$.

The efficiency of an IS is measured by its efficiency index (*EI*), expressed as $EI = \varphi^{\frac{1}{T}}$, where $\varphi$ is the convergence order and T is the sum of all distinct function evaluations in one iterative round. Consequently, the EI of the IS in equations (3) and (4) is 1.4142. To contribute to the development of derivative-free IS with high convergence orders and efficiency indices, new and effective derivative-free families of fourth, seventh, and eighth-order convergence IS are proposed in this manuscript. These IS are modifications of the SIS developed via the application of divided difference, weight function, and composition techniques.

The manuscript is structured as follows: Section 2 presents the development of the IS, Section 3 establishes the convergence of the IS, Section 4 discusses the implementation of the developed IS, and Section 5 concludes the manuscript.

## 2.0 METHODOLOGY

### 2.1 The iterative scheme

Consider the modification of the NIS put forward in Alfio et al.,(2000) with IS as

$$y_n = w_n - \frac{\Gamma(w_n)}{\Gamma'(w_n)},$$

$$w_{n+1} = y_n - \frac{\Gamma(y_n)}{\Gamma'(w_n)}. \tag{5}$$

To annihilate the function derivative in (5), we used the approximation $\Gamma'(w_n) \approx \varphi(w_n)$ so that

$$\Gamma'(w_n) \approx \frac{\Gamma\left(w_n + \delta(\Gamma(w_n))^m\right) - \Gamma(w_n)}{\delta(\Gamma(w_n))^m} = \varphi(w_n) \tag{6}$$

where $m \geq 2$. Consider the introduction of a real-value weight function (RVWF) $H(t)$ $\left(\text{where } t = \frac{\Gamma(y)}{\Gamma(x)}\right)$ to the second step of (5) such that the Taylor series expansion of $H(t)$ about 1 is

$$B(t) = B(1) + \sum_1^4 \frac{1}{i} B^{(i)}(1)(t-1)^i, \tag{7}$$

where $B^{(i)}$ is the ith derivatives of the real-value function (RVF) evaluated at 1.

If we take $m = 3$ and for $\delta \geq 0$, $\delta \in (0,1]$ in (6), then the following derivative free IS is put forward as:.

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)},$$

$$w_{n+1} = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)}. \tag{8}$$

Our claim is that, for some suitable values for $B(1), B'(1), B''(1), B'''(1)$ and $B^{(iv)}(1)$, the IS in (8) will produce a family of optimal CO 4 that require evaluation of three functions in an iteration cycle and in the sense of Kung-Traub (1974), has a generic efficiency index $EI = 1.1587$. This claim will be established in the proof of Theorem 3.1 in the next section.

To enhance the IS (8) so as to improve its CO and EI, an additional iterative step with two new RVWF $G(v)$ and $H(t)$ (where $v = \frac{\Gamma(z)}{\Gamma(y)}$) with Taylor series expansion around 0 as

$$H(t) = H(0) + \sum_1^4 \frac{1}{i} H^{(i)}(0)t^i \tag{9}$$

and

$$G(v) = G(0) + \sum_1^4 \frac{1}{i} G^{(i)}(0)v^i, \tag{10}$$

respectively, are introduced to the IS in (8). Consequently, a new IS is obtained as

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)},$$

$$z_n = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)},$$

$$w_{n+1} = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)}\{G(v_n) \times H(t_n)\}. \tag{11}$$

The IS (11) require four distinct functions evaluation in one complete iteration and with some suitable conditions imposed on the RVWF $G(v_n)$ and $H(t_n)$ and their derivatives, can attained CO of seven. For this reason, any concrete member of the IS in (11) will have $EI = 1.6266$. Theorem 3.2 provides the conditions for which the IS in (11) attains convergence with order seven.

Again, to improve the IS in (11) in terms of its CO and efficiency, an additional RVWF $Q(s)$, $s = \frac{\Gamma(z)}{\Gamma(w)}$, with its Taylor series expansion about 0 as

$$Q(s) = Q(0) + \sum_1^4 \frac{1}{i} Q^{(i)}(0)s^i, \tag{12}$$

is introduced to the IS third step. We note here that, for suitable conditions imposed on the RVWF, a CO eight IS that is optimal is obtained as:

$$y_n = w_n - \frac{\Gamma(w_n)}{\Gamma'(w_n)},$$

$$z_n = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)},$$

$$w_{n+1} = z_n - B(t_n)\frac{\Gamma(z_n)}{\varphi(w_n)}\{G(v_n) \times H(t_n)\} \times Q(s_n). \tag{13}$$

If the convergence of the family of IS in (13) is established, then any concrete member of it shall have $EI = 1.6818$. Its conditions for convergence with order eight is provided in Theorem 3.

## 2.2 Test for the Iterative Structures Convergence

We acknowledged the definitions below while proving the theorems that establishes the convergence of all the iterative structures put forward.

***Definition 1:*** Assume that $d_n = w_n - \theta$ be the IS error at the nth iteration and if an equation of the form $d_{n+1} = \zeta d_n^{\varphi} + O(d_n^{\varphi+1})$ can be deduced from the IS via the Taylor series expansions on the functions $\Gamma(\cdot)$ it contains, then $d_{n+1}$its called the error equation, $\zeta$ the error constant and $\varphi$ is CO of the IS.

A more detailed illustration on Definition 1 can be found in the proof of several IS that appear in (Ogbereyivwe and Ojo-Orobosa (2021a, 2021b); Ogbereyivwe and Izevbizua (2023); Ogbereyivwe and Umar (2023)) and reference therein.

***Definition 2:*** Let the equation $d_{n+1} = \zeta d_n^{\varphi} + O(d_n^{\varphi+1})$ holds for an IS as posited in Definition 1, then the measure of efficiency of an IS and referred to as Efficiency index is given by the value $EI = \varphi^{\frac{1}{T}}$ (where T is the number of all distinct functions $\Gamma(\cdot)$ required to be evaluated in an IS cycle).

***Definition 3:*** Let $\Gamma(w_{n-1})$, $\Gamma(w_n)$ and $\Gamma(w_{n+1})$ be the values of function $\Gamma(\cdot)$ evaluated at the iteration points $w_{n-1}, w_n$ and $w_{n+1}$, then the quantity $\varphi_{coc}$ of an IS is approximated by Jay in Jay(2001) as

$$\delta_{coc} \approx \frac{\ln|\Gamma(w_{n+1})/\Gamma(w_n)|}{\ln|\Gamma(w_n)/\Gamma(w_{n-1})|}. \tag{14}$$

Next, is the convergence test for the developed IS, conducted in MATHEMATICAL 9.0 software environment. Programs were designed based on the claims in the theorems and implemented for convergence.

**Theorem 1:** Let $\Gamma: I \subset R \to R$ be a scalar function that is sufficiently differentiable on the interval $I$ with a simple solution $\theta \in I$. If $w_0$ close to $\theta$ is chosen and used in (8), then the sequence $\{w_n\}_{n\geq 0}, (w_n \in I)$ of iterative outputs produced by the family of IS in (8) converges to $\theta$ with CO four when $B(t)$ is imposed with the conditions $B(1) = 1$, $B'(1) = -2$, $B''(1) = -4$ and $B'''(1) = 0$.

*Proof.* Let $c_n = \frac{\Gamma^{(j)}(\theta)}{j!\,\Gamma'(\theta)}$, $j \geq 2$ and $w = w_n$ in the Taylor series expansion of $\Gamma(w)$ and $\Gamma'(w)$. For easy documentation in the MATHEMATICA software environment, we let $\Gamma w$, $\Gamma A$, $\Gamma y$ to represent $\Gamma(w)$, $\Gamma(A)$, $\Gamma(y)$ respectively. And the convergence test program is put forward as:

$\Gamma w = (d + c_2 * d^2 + c_3 * d^3 + c_4 * d^4 + c_5 * d^5 + c_6 * d^6 + c_7 * d^7 + c_8 * d^8);$
$A = d + \delta(\Gamma w)^3;$
$\Gamma A = (A + c_2 * A^2 + c_3 * A^3 + c_4 * A^4 + c_5 * A^5 + c_6 * A^6 + c_7 * A^7 + c_8 * A^8);$

$M = \text{Series}\left[\left(\frac{\Gamma A - \Gamma w}{A - d}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
$y = \text{Series}\left[e - \left(\frac{\Gamma w}{M}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
$\Gamma y = (y + c_2 * d^2 + c_3 * d^3 + c_4 * d^4);$
$t = \text{Series}\left[\left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
$B = \text{Series}[1 - 2 * (t - 1) - 2 * (t - 1)^2, \{d, 0{,}4\}] //\text{Simplify};$
$w = \text{Series}\left[y - B * \left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
The convergence test program above, produced an equation as:

$$w_{n+1} = \theta + \left(7c_2^2 - c_2 c_3 + \frac{c_2 c_5}{\delta}\right) d_2^4 + O(d_n^5). \tag{15}$$

By Definition 1, the expression in (15) is the IS in (8) error equation when the conditions in Theorem 1 are imposed on the parameters. Consequently, the IS in (8) is of CO four.

***Remark 1:*** For any RVWF $B(t)$ satisfying the conditions $B(1) = 1$, $B'(1) = 2$, $B''(1) = 4$ and $B'''(0) = 0$, the IS in (8) produces a CO four. For instance, choose $B(t) = 1 - 2(t - 1) - 2(t - 1)^2$ and we have a new IS (OA4) as:

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)}$$
$$w_{n+1} = y_n - \left[1 - 2\left(\frac{\Gamma(y_n)}{\Gamma(w_n)} - 1\right) - 2\left(\frac{\Gamma(y_n)}{\Gamma(w_n)} - 1\right)^2\right]\frac{\Gamma(y_n)}{\varphi(w_n)}. \tag{16}$$

**Theorem 2:** Assume the conditions imposed on $\Gamma(w)$ holds as in Theorem 1, then the sequence $\{w_n\}_{n\geq 0}, (w_n \in I)$ of iterative outputs produced by the family of IS in (11) converges to $\theta$ with CO seven when $B(t)$, $H(t)$ and $G(v)$ are imposed with the conditions $B(1) = G(0) = H(0) = 1$, $B'(1) = -2$, $B''(1) = -4$ and $B'''(0) = 0$, $G''(0) < \infty$, $H'(0) = 0$, $H''(0) = 2$, $H'''(0) = 24$, $H^{(iv)}(1) < \infty$.

*Proof:* Here, the procedure used in the proof of Theorem 1 is followed. For the sake of the program, let $\Gamma z$ be $\Gamma(z)$. The program for the convergence test is put forward next:

$\Gamma w = (d + c_2 * d^2 + c_3 * d^3 + c_4 * d^4 + c_5 * d^5 + c_6 * d^6 + c_7 * d^7 + c_8 * d^8);$
$A = d + \delta(\Gamma w)^3;$
$\Gamma A = (A + c_2 * A^2 + c_3 * A^3 + c_4 * A^4 + c_5 * A^5 + c_6 * A^6 + c_7 * A^7 + c_8 * A^8);$
$M = \text{Series}\left[\left(\frac{\Gamma A - \Gamma w}{A - d}\right), \{d, 0{,}7\}\right] //\text{Simplify};$
$y = \text{Series}\left[e - \left(\frac{\Gamma w}{M}\right), \{d, 0{,}7\}\right] //\text{Simplify};$
$\Gamma y = (y + c_2 * d^2 + c_3 * d^3 + c_4 * d^4);$
$t = \text{Series}\left[\left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0{,}7\}\right] //\text{Simplify};$

$B = \text{Series}[1 - 2 * (t - 1) - 2 * (t - 1)^2, \{d, 0{,}7\}]//\text{Simplify};$
$z = \text{Series}\left[y - B * \left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0{,}4\}\right]//\text{Simplify};$
$\Gamma z = \text{Series}[(z + c_2 * z^2 + c_3 * z^3 + c_4 * z^4), \{d, 0{,}4\}]//\text{Simplify};$
$v = \text{Series}\left[\left(\frac{\Gamma z}{\Gamma y}\right), \{d, 0{,}7\}\right]//\text{Simplify};$
$G = \text{Series}[1 + v, \{d, 0{,}7\}]//\text{Simplify};$
$H = \text{Series}[1 + t^2 + 4 * t^3, \{d, 0{,}7\}]//\text{Simplify};$
$w = \text{Series}\left[z - B * \left(\frac{\Gamma z}{M}\right) * H * G, \{d, 0{,}7\}\right]//\text{Simplify}$
The program output is:

$$w_{n+1} = \theta + 2c_2^2(-7c_2^2 + c_3)d_n^7 + O(e_n^8). \quad (17)$$

Again, by Definition 1, the IS in (9) has the equation in (17) as error equation. As a result of this, its CO is seven. This verified the claim in Theorem 2.

***Remark 2:*** For some RVWF $B(t)$, $G(v)$ and $H(t)$ jointly satisfying the conditions in Theorem 2, will produce a CO seven IS. Suppose $B(t) = 1 - 2(t - 1) - 2(t - 1)^2$, $G(v) = 1 + v$ and $H(t) = 1 + t^2 + 4t^3$, $\alpha \in R$, we obtain a new order seven IS( OA7) given as:

$$\begin{aligned} y_n &= w_n - \frac{\Gamma(w_n)}{\varphi(w_n)}; \\ z_n &= y_n - B(t)\frac{\Gamma(y_n)}{\varphi(w_n)}; \\ w_{n+1} &= z_n - B(t)\frac{\Gamma(z_n)}{\varphi(w_n)}\left[1 + \frac{\Gamma(z_n)}{\Gamma(y_n)}\right]\left[1 + \left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^2 + 4\left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^3\right]. \end{aligned} \quad (18)$$

**Theorem 3:** Consider the conditions imposed on $\Gamma(w)$ in Theorem 1 holds, then for any $w_0$ close to $\theta$, a sequence $\{w_n\}_{n\geq 0}, (w_n \in I)$ of iterative outputs produced by the family of IS in (13) converges to $\theta$ with CO eight when $B(t)$, $G(v)$, $H(t)$ and $Q(s)$ is imposed with the conditions that $B(1) = G(0) =, G'(0) = H(0) = Q(0) = 1$, $B'(1) = -2$, $B''(1) = -4$, $B'''(0) = 0$, $H'(0) = 0, Q'(0) = 2$, $H''(0) = -20$, $Q''(0) < \infty$.
*Proof:* The convergence error is obtained using the program below:
$\Gamma w = (d + c_2 * d^2 + c_3 * d^3 + c_4 * d^4 + c_5 * d^5 + c_6 * d^6 + c_7 * d^7 + c_8 * d^8);$
$A = d + \delta(\Gamma w)^3;$
$\Gamma A = (A + c_2 * A^2 + c_3 * A^3 + c_4 * A^4 + c_5 * A^5 + c_6 * A^6 + c_7 * A^7 + c_8 * A^8);$
$M = \text{Series}\left[\left(\frac{\Gamma A - \Gamma w}{A - d}\right), \{d, 0{,}8\}\right]//\text{Simplify};$
$y = \text{Series}\left[d - \left(\frac{\Gamma w}{M}\right), \{d, 0{,}8\}\right]//\text{Simplify};$
$\Gamma y = (y + c_2 * d^2 + c_3 * d^3 + c_4 * d^4);$
$t = \text{Series}\left[\left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0{,}8\}\right]//\text{Simplify};$
$B = \text{Series}[1 - 2 * (t - 1) - 2 * (t - 1)^2, \{d, 0{,}8\}]//\text{Simplify};$

$$z = \text{Series}\left[y - B * \left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0,8\}\right] //\text{Simplify};$$
$$\Gamma z = \text{Series}[(z + c_2 * z^2 + c_3 * z^3 + c_4 * z^4), \{d, 0,8\}] //\text{Simplify};$$
$$v = \text{Series}\left[\left(\frac{\Gamma z}{\Gamma y}\right), \{d, 0,8\}\right] //\text{Simplify};$$
$$G = \text{Series}[1 + v, \{d, 0,8\}] //\text{Simplify};$$
$$H = \text{Series}[1 + t^2 - 10 * t^3, \{d, 0,8\}] //\text{Simplify};$$
$$s = \text{Series}\left[\left(\frac{\Gamma z}{\Gamma w}\right), \{d, 0,8\}\right] //\text{Simplify};$$
$$Q = \text{Series}[1 + 2 * s + Q''(0) * s^2, \{d, 0,8\}] //\text{Simplify};$$
$$w = \text{Series}\left[z - B * \left(\frac{\Gamma z}{M}\right) * H * G * Q, \{d, 0,8\}\right] //\text{Simplify}$$

The program output is:

$$w_{n+1} = \theta + \frac{1}{24}[c_2(7c_2^2 - c_3)(\delta c_2^2 + 64c_2^4 - 18c_2^2c_3 + c_3^2 + c_2c_4)]d_n^8 + O(e_n^9). \quad (19)$$

The equation in (19) is the IS error equation. From the error equation given in Definition 1, it can be deduced from the expression in (19), that the family of IS in (11) has a general CO of eight.

***Remark 3:*** For RVWF $B(t), G(t)$, $G(t)$ and $G(s)$ satisfying the conditions in Theorem 3 will produces a CO eight. Suppose $B(t) = 1 - 2(t-1) - 2(t-1)^2$, $G(v) = 1 + v$ and $H(t) = 1 + t^2 - 10t^3$, $Q(s) = 1 + 2s + \tau s^2$, $\tau \in R$ we obtain the new IS OA8 given as:

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)};$$
$$z_n = y_n - B(t)\frac{\Gamma(y_n)}{\varphi(w_n)};$$
$$w_{n+1} = z_n - B(t)\frac{\Gamma(z_n)}{\varphi(w_n)}\begin{bmatrix}1 + \left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^2 \\ -10\left(\frac{\Gamma(z_n)}{\Gamma(w_n)}\right)^3\end{bmatrix}\left[1 + \frac{\Gamma(z_n)}{\Gamma(y_n)}\right]\begin{bmatrix}1 + 2\left(\frac{\Gamma(z_n)}{\Gamma(w_n)}\right) \\ +\tau\left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^2\end{bmatrix}. \quad (20)$$

## 3.0 RESULTS

The implementation and applicability of the developed IS (OA4, OA7, and OA8) are illustrated in this section. Computation programs for OA4, OA7, OA8a (using $\tau = 0$ in Eq. (20) ) and OA8b (using $\tau = 12$ in Eq. (20) ) were written in MAPLE 9 environment and used to locate the solution of some NL equations. Their computation outputs were juxtaposed with that of some existing IS (with same CO) outputs (see Tables 1 and 2). The compared IS includes the CO four Kung-Traub (1974) (KT4), CO seven IS presented in Panday and Jaiswal (2017) (PJ7), CO eight IS in Soleymani (2011) (SL8) and the CO eight IS of Yasmin et al.,(2016) (YZS8).

The metric used for contrast includes, the IS required number of iteration rounds to attain convergence (IT), residual function of last iteration $|f(w_{n+1})|$ and computational order of convergence $\varphi_{coc}$ defined in Jay(2001) (see Definition 3.3).

The precision digits and error bound used in the programing are 2000 significant figures and $\varepsilon = 10^{-1000}$ respectively. The NL equations used for the test are:

$$\Gamma_1(w) = \sqrt{w^2 + 2w + 5} - 2\sin w + w^2 + 3\,, \quad \theta = 2.331967\ldots$$

$$\Gamma_2(w) = e^{(w^2 - w)} - \cos(1 - w^2) + w^3 + 1\,, \quad \theta = -1$$

Table 1. Iterative schemes results comparison for $\Gamma_1(w)$ and $\Gamma_2(w)$ .

| $\Gamma_n(w)$ | IS | $w_0$ | IT | $\lvert\Gamma(w_{n+1})\rvert$ | $\varphi_{coc}$ |
|---|---|---|---|---|---|
| $\Gamma_1(w)$ | $KT4_{\delta=.001}$ | 1.5 | 5 | 6.8894e-1193 | 4 |
| | $OA4_{\delta=.001}$ | | 5 | 5.3424e-1235 | 4 |
| | PJ7 | | 4 | 1.1103e-0431 | 7 |
| | $OA7_{\delta=.001}$ | | 3 | 3.3332e-0377 | 7 |
| | SL7 | | 4 | 8.4289e-1842 | 7 |
| | YZS8 | | Failed | - | - |
| | $OA8a_{\delta=.001}$ | | 3 | 7.5523e-0493 | 8 |
| | $OA8b_{\delta=.001}$ | | 3 | 7.4711e-0493 | 8 |
| $\Gamma_2(w)$ | $KT4_{\delta=.0001}$ | 0.4 | 7 | 9.1016e-0857 | 4 |
| | $OA4_{\delta=.0001}$ | | 6 | 2.0764e-1511 | 4 |
| | PJ7 | | Failed | - | - |
| | $OA7_{\delta=.0001}$ | | 5 | 3.2173e-1488 | 7 |
| | SL7 | | 5 | 7.5632e-0765 | 7 |
| | YZS8 | | Failed | - | - |
| | $OA8a_{\delta=.0001}$ | | 4 | 1.6975e-0465 | 8 |
| | $OA8b_{\delta=.0001}$ | | 4 | 2.8023e-0453 | 8 |

Next, the IS were applied to solve some real life models.

***Model 1:*** Environmental engineering model (Chapra and Canale, 2020)

Consider the equation describing the downstream oxygen level C(mg/L) when sewage is discharged in a river given as

$$C = 10 - 20\,(\exp(-.15w) - \exp(-.5w))$$

where w (in kilometers) represents distance downstream that is desired to be known. A problem may arise to determine the distance downstream (w) when the oxygen level falls to 5mg/L. In this case, the above model becomes

$$\Gamma_3(w) = 20\,(\exp(-.15w) - \exp(-.5w)) - 5,$$

and its real solution is $w = 0.9762 \ldots$

***Model 2:*** Aerospace engineering model (Kallrath, 2002).

Consider the Kepler's model in astronomy given as

$$M = w - e\sin w\,, \quad M \in [0,1],$$

where M and e are mean anomaly and e eccentricity respectively and play a significant role in the Kepler's model. The eccentric anomaly w often used to determine the position of a moving point in a Keplerian orbit. A case may require to determine the eccentric anomaly w when $M = 0.6$ and $e = 0.9$. In the case, the Kepler's model become:

$$\Gamma_4(w) = w - 0.9\sin w - 0.6.$$

The actual solution of the above equation is $w = 1.4975 \ldots$

The computation results when the developed IS and compared schemes are presented in Table 2.

### 3.1 Results Discussion

The symbol $IS_\delta$ used in Table 4.1-4.2, denotes an Iterative Structure when the value of the parameter $\delta$ as indicated is used. The computation outputs are presented in tables in the form $De - E$ to implies $D \times 10^E$ where $D, E \in R$. From Table 1 and Table 2, it can be observed that our contributed IS (OA4, OA7,OA8a and OA8b) at varied values of parameters, solved all the tested examples with competitive level of precision. In fact, in Table 1 where the compared IS PJ7 and YZS8 failed to provide solutions, all the IS introduced in this work, obtained the solutions with high convergence. The implication is that our IS can be used where existing IS failed to solve NL models. Finally, on all the problem tested, the $\varphi_{coc}$ produced by the various developed IS (see the last column of Table 1 and Table 2) coincided with their respective convergence order values obtained from the theoretical analysis in Section 3.

**Table 2**. Iterative schemes results comparison for $\Gamma_3(w)$ and $\Gamma_4(w)$ .

| $\Gamma_n(w)$ | IS | $w_0$ | IT | $\lvert\Gamma(w_{n+1})\rvert$ | $\varphi_{coc}$ |
|---|---|---|---|---|---|
| $\Gamma_3(w)$ | $KT4_{\delta=-.001}$ | 1.2 | 5 | 1.5185e-1013 | 4 |
| | $OA4_{\delta=-.001}$ | | 5 | 1.3414e-0766 | 4 |
| | PJ7 | | 4 | 5.0996e-0750 | 7 |
| | $OA7_{\delta=-.001}$ | | 4 | 3.4046e-1825 | 7 |
| | SL7 | | 4 | 4.9214e-0840 | 7 |
| | YZS8 | | 3 | 4.7383e-0246 | 8 |
| | $OA8a_{\delta=-.001}$ | | 3 | 3.2424e-0326 | 8 |
| | $OA8b_{\delta=-.001}$ | | 3 | 4.7633e-0329 | 8 |
| | $KT4_{\delta=1}$ | | 6 | 4.0530e-1579 | 4 |
| | $OA4_{\delta=1}$ | | 5 | 3.6579e-0673 | 4 |
| | PJ7 | | 4 | 2.3479e-1579 | 7 |

| | | | | | |
|---|---|---|---|---|---|
| $\Gamma_4(w)$ | $OA7_{\delta=1}$ | 0.4 | 4 | 1.9707e-0933 | 7 |
| | SL7 | | 4 | 1.0886e-0727 | 7 |
| | YZS8 | | 3 | 2.0746e-0370 | 8 |
| | $OA8a_{\delta=1}$ | | 4 | 2.3025e-1117 | 8 |
| | $OA8b_{\delta=1}$ | | 4 | 2.6170e-1273 | 8 |

## 4.0 CONCLUSION

We applied three numerical techniques - divided difference, weight function, and composition - to modify the two-step iterative structure presented in Alfio et al. (2000). This modification yielded novel, optimal iterative schemes with convergence orders of 4, 7, and 8 for solving nonlinear equations. The computational performance of the proposed iterative structures was evaluated on real-life problems, yielding efficient solutions and confirming their ability as a numerical tool for analyzing nonlinear sustainability models.

A promising avenue for future research directions include exploring the complex dynamics and chaotic behavior of the proposed iterative schemes, which presents a fertile area for further investigation

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