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BEHIND AN ITERATIVE SCHEME: IS THE ITERATIVE SCHEME A TOOL FOR SUSTAINABLE DEVELOPMENT?

O. OGBEREYIVWE[1], E. B. IDISI[2] AND O. D. AGBEDEYI[3]

[1]Department of Mathematics, Delta State University of Science and Technology, Ozoro.
[2]Department of Environmental Management, Delta State University of Science and Technology, Ozoro.
[3]Department of Mathematics and Statistics, Delta State Polytechnic, Ogwashi-Uku.
Corresponding Author: ogho2015@gmail.com
[2]idisieb@dsust.edu.ng , [3]desmondagbedeyi@gmail.com

ABSTRACT: Contrary to the thought that Mathematics and Sustainable Development are unconnected, Mathematical tools are veritable means of solving complex nonlinear models that arises from the field of science, environment, economics and engineering. Obtaining the solution of nonlinear models is key and necessary to evaluate sustainability in quantitative terms. But, some of these nonlinear models are unsolvable via analytic means, hence numerical iterative procedures are utilised. However, the efficiency of an iterative procedure for solving mathematical models is a function of time and cost, which are key factors that cannot be undermined in achieving Sustainable Development. One golden rule in the development of new iterative schemes for solving nonlinear models, is that it should be of low computation cost. The presence of functions derivatives evaluations in an iterative scheme no doubt, increases its implementation cost. More so, obtaining derivative of some complex functions can be big challenge. In this regard, three class of fourth, seventh and eight-order derivative free iterative structures are developed and put forward in this work. The convergence test on the developed iterative scheme were established using some designed computer programs that are executed in MATHEMATICA software environment. To verify the schemes as tools for sustainable economic development, some recently modelled physical phenomena were solved using the developed iterative schemes and their results were placed in juxtaposition with that of some existing contemporary schemes.

INTRODUCTION

Sustainable Development practice satisfies the present needs, without jeopardizing future generation ability to satisfy their needs (Soumen, 2018). Mathematical modelling are veritable tools used for understanding, prediction and control of development process toward achieving sustainable development. Consider the environmental management mathematical model of the Ehrlich and Holdren postulation on the society negative impact on its environment and put forward in Ehrlich and Holdren (1971) as

$$\mathrm{I} = \mathrm{P} \times \mathrm{A} \times \mathrm{T}, \tag{1}$$

where the parameters P, A and T represents population, affluence and technology respectively. If the population (P), affluence (A) and technology (T) are expressed as functions of time (w), then the right part of the equation in (1) becomes a nonlinear equation in terms of time (w). A problem may arise to determine the time (w), for which the impact of the society activities will be negligible so as assure that the next generations will still enjoy quality lives. In this case, the equation in (1) can be generally expressed as

$$\Gamma(\mathrm{w}) = 0. \tag{2}$$

One fundamental and daunting task in computational sciences is that of locating the solution θ of nonlinear (NL) equations in (2), where $\Gamma: \mathrm{R} \to \mathrm{R}$. For NL algebraic equations that are of degree up to four, certain analytical methods exist for locating their solutions. However, it is a huge challenge to determine solution of a NL algebraic equation of degree more than four or NL equations that involves mix of one or more of transcendental, trigonometric, logarithmic or exponential functions. In this case, Iterative Structures (IS) were designed and put forward for locating the solution of NL equations. The IS technique involves a recurrent formula used in approximating θ of the target NL equation such that, for every complete assessment of the recurrent cycle, an improved estimation of θ is continued to be obtained until the exact value of θ is achieved.

Fundamentally, the IS for solving NL equations can be classified based on functions evaluation, into two categories. The first category requires the evaluation of functions $\Gamma(\cdot)$ and their derivative(s) $\Gamma'(\cdot)$ (see Ogbereyivwe and Ojo-Orobosa (2021); Ogbereyivwe and Izevbizua (2022); Padilla et al.,(2022) and reference there in) and the second category are the IS that are derivative free (Soleymani (2010); Sivakumar et al., (2021); Bahgat (2021); Zafar (2016); Panday and Jaiswal (2017)). Because the evaluation of functions derivatives incurs additional computational cost and the fact that obtaining derivative of some functions are complex, had resulted in the development of IS that are without derivatives. For instance, by annihilating the first derivative $\Gamma'(\mathrm{w})$ in classical Newton iterative structure (NIS) given as

$$\mathrm{w}_{\mathrm{n}+1} = \mathrm{w}_{\mathrm{n}} - \frac{\Gamma(\mathrm{w}_{\mathrm{n}})}{\Gamma'(\mathrm{w}_{\mathrm{n}})}, \quad \mathrm{n} = 0,1,2,\ldots \tag{3}$$

using the divided difference technique to estimate $\Gamma'(w)$, the classical derivative free Steffensen iterative structure (SIS) (Steffensen,1933) is obtained as:

$$w_{n+1} = w_n - \frac{(\Gamma(w_n))^2}{\Gamma(w_n+\Gamma(w_n))-\Gamma(w_n)}. \tag{4}$$

The method in (4) is of convergence order (CO) two and because it require the assessment of two functions in an iteration round, it conserved the optimality property of the NIS (3) in the view of Kung-Traub conjecture, see Kung and Traub (1974). It was conjectured by Kung and Traub that, an IS that consumes n-functions assessment in an iteration round is optimal when it achieve CO bound of 2^{n-1}. On the other hand, the efficiency of an IS is measured by a quantity referred to as efficiency index (EI) expressed as $\varphi^{\frac{1}{T}}$, where φ is the CO and T sum of all distinct function assessment in one iterative round of the IS. In this case, the EI of the IS in (3) and (4) is 1.4142.

To contribute to the development of derivative free IS with high CO and efficiency index, some new and effective derivative free families of fourth, seventh and eight-order convergence IS that are modifications of the SIS developed via the application of the divided difference, weight function (WF) and composition techniques are put forward in this manuscript. The structural layout of this paper is described as following. Section 2 is dedicated to the IS development, Section 3 is for the establishment of the IS convergence, Section 4 treated the implementation of the developed IS while Section 5 is conclusion.

2.0 METHODOLOGY

2.1 The iterative scheme

Consider the modification of the NIS put forward in Alfio et al.,(2000) with IS as

$$y_n = w_n - \frac{\Gamma(w_n)}{\Gamma'(w_n)},$$

$$w_{n+1} = y_n - \frac{\Gamma(y_n)}{\Gamma'(w_n)}. \tag{5}$$

To annihilate the function derivative in (5), we used the approximation $\Gamma'(w_n) \approx \varphi(w_n)$ so that

$$\Gamma'(w_n) \approx \frac{\Gamma\left(w_n+\delta(\Gamma(w_n))^m\right)-\Gamma(w_n)}{\delta(\Gamma(w_n))^m} = \varphi(w_n) \tag{6}$$

where $m \geq 2$. Consider the introduction of a real-value weight function (RVWF) $H(t)$ $\left(\text{where } t = \frac{\Gamma(y)}{\Gamma(x)}\right)$ to the second step of (5) such that the Taylor series expansion of $H(t)$ about 1 is

$$B(t) = B(1) + \sum_1^4 \frac{1}{i} B^{(i)}(1)(t-1)^i, \quad (7)$$

where $B^{(i)}$ is the ith derivatives of the real-value function (RVF) evaluated at 1.

If we take $m = 3$ and for $\delta \geq 0,\ \delta \in (0,1]$ in (6), then the following derivative free IS is put forward as:.

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)},$$

$$w_{n+1} = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)}. \quad (8)$$

Our claim is that, for some suitable values for $B(1), B'(1), B''(1), B'''(1)$ and $B^{(iv)}(1)$, the IS in (8) will produce a family of optimal CO 4 that require evaluation of three functions in an iteration cycle and in the sense of Kung-Traub (1974), has a generic efficiency index $EI = 1.1587$. This claim will be established in the proof of Theorem 3.1 in the next section.

To enhance the IS (8) so as to improve its CO and EI, an additional iterative step with two new RVWF $G(v)$ and $H(t)$ (where $v = \frac{\Gamma(z)}{\Gamma(y)}$) with Taylor series expansion around 0 as

$$H(t) = H(0) + \sum_1^4 \frac{1}{i} H^{(i)}(0)t^i \quad (9)$$

and

$$G(v) = G(0) + \sum_1^4 \frac{1}{i} G^{(i)}(0)v^i, \quad (10)$$

respectively, are introduced to the IS in (8). Consequently, a new IS is obtained as

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)},$$

$$z_n = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)},$$

$$w_{n+1} = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)}\{G(v_n) \times H(t_n)\}. \quad (11)$$

The IS (11) require four distinct functions evaluation in one complete iteration and with some suitable conditions imposed on the RVWF $G(v_n)$ and $H(t_n)$ and their derivatives, can attained CO of seven. For this reason, any concrete member of the IS in (11) will have $EI = 1.6266$. Theorem 3.2 provides the conditions for which the IS in (11) attains convergence with order seven.

Again, to improve the IS in (11) in terms of its CO and efficiency, an additional RVWF $Q(s)$, $s = \frac{\Gamma(z)}{\Gamma(w)}$, with its Taylor series expansion about 0 as

$$Q(s) = Q(0) + \sum_1^4 \frac{1}{i} Q^{(i)}(0)s^i, \qquad (12)$$

is introduced to the IS third step. We note here that, for suitable conditions imposed on the RVWF, a CO eight IS that is optimal is obtained as:

$$y_n = w_n - \frac{\Gamma(w_n)}{\Gamma'(w_n)},$$

$$z_n = y_n - B(t_n)\frac{\Gamma(y_n)}{\varphi(w_n)},$$

$$w_{n+1} = z_n - B(t_n)\frac{\Gamma(z_n)}{\varphi(w_n)}\{G(v_n) \times H(t_n)\} \times Q(s_n). \qquad (13)$$

If the convergence of the family of IS in (13) is established, then any concrete member of it shall have $EI = 1.6818$. Its conditions for convergence with order eight is provided in Theorem 3.

2.2 Test for the Iterative Structures Convergence

We acknowledged the definitions below while proving the theorems that establishes the convergence of all the iterative structures put forward.

Definition 1: Assume that $d_n = w_n - \theta$ be the IS error at the nth iteration and if an equation of the form $d_{n+1} = \zeta d_n^{\varphi} + O(d_n^{\varphi+1})$ can be deduced from the IS via the Taylor series expansions on the functions $\Gamma(\cdot)$ it contains, then d_{n+1}its called the error equation, ζ the error constant and φ is CO of the IS.

A more detailed illustration on Definition 1 can be found in the proof of several IS that appear in (Ogbereyivwe and Ojo-Orobosa (2021); Ogbereyivwe and Izevbizua (2023); Ogbereyivwe and Umar (2023)) and reference therein.

Definition 2: Let the equation $d_{n+1} = \zeta d_n^{\varphi} + O(d_n^{\varphi+1})$ holds for an IS as posited in Definition 1, then the measure of efficiency of an IS and referred to as Efficiency index is given by the value $EI = \varphi^{\frac{1}{T}}$ (where T is the number of all distinct functions $\Gamma(\cdot)$ required to be evaluated in an IS cycle).

Definition 3: Let $\Gamma(w_{n-1})$, $\Gamma(w_n)$ and $\Gamma(w_{n+1})$ be the values of function $\Gamma(\cdot)$ evaluated at the iteration points w_{n-1}, w_n and w_{n+1}, then the quantity φ_{coc} of an IS is approximated by Jay in Jay(2001) as

$$\delta_{coc} \approx \frac{\ln|\Gamma(w_{n+1})/\Gamma(w_n)|}{\ln|\Gamma(w_n)/\Gamma(w_{n-1})|}. \tag{14}$$

Next, is the convergence test for the developed IS, conducted in MATHEMATICAL 9.0 software environment. Programs were designed based on the claims in the theorems and implemented for convergence.

Theorem 1: Let $\Gamma: I \subset R \to R$ be a scalar function that is sufficiently differentiable on the interval I with a simple solution $\theta \in I$. If w_0 close to θ is chosen and used in (8), then the sequence $\{w_n\}_{n\geq 0}, (w_n \in I)$ of iterative outputs produced by the family of IS in (8) converges to θ with CO four when $B(t)$ is imposed with the conditions $B(1) = 1, B'(1) = -2,\ B''(1) = -4$ and $B'''(1) = 0$.

Proof. Let $c_n = \frac{\Gamma^{(j)}(\theta)}{j!\,\Gamma'(\theta)},\ j \geq 2$ and $w = w_n$ in the Taylor series expansion of $\Gamma(w)$ and $\Gamma'(w)$. For easy documentation in the MATHEMATICA software environment, we let Γw, ΓA, Γy to represent $\Gamma(w)$, $\Gamma(A)$, $\Gamma(y)$ respectively. And the convergence test program is put forward as:

$\Gamma w = (d + c_2 * d^2 + c_3 * d^3 + c_4 * d^4 + c_5 * d^5 + c_6 * d^6 + c_7 * d^7 + c_8 * d^8);$
$A = d + \delta(\Gamma w)^3;$
$\Gamma A = (A + c_2 * A^2 + c_3 * A^3 + c_4 * A^4 + c_5 * A^5 + c_6 * A^6 + c_7 * A^7 + c_8 * A^8);$
$M = \text{Series}\left[\left(\frac{\Gamma A - \Gamma w}{A - d}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
$y = \text{Series}\left[e - \left(\frac{\Gamma w}{M}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
$\Gamma y = (y + c_2 * d^2 + c_3 * d^3 + c_4 * d^4);$
$t = \text{Series}\left[\left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
$B = \text{Series}[1 - 2 * (t - 1) - 2 * (t - 1)^2, \{d, 0{,}4\}]//\text{Simplify};$
$w = \text{Series}\left[y - B * \left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0{,}4\}\right] //\text{Simplify};$
The convergence test program above, produced an equation as:

$$w_{n+1} = \theta + \left(7c_2^2 - c_2c_3 + \frac{c_2c_5}{\delta}\right) d_2^4 + O(d_n^5). \tag{15}$$

By Definition 1, the expression in (15) is the IS in (8) error equation when the conditions in Theorem 1 are imposed on the parameters. Consequently, the IS in (8) is of CO four.

Remark 1

For any RVWF $B(t)$ satisfying the conditions $B(1) = 1,\ B'(1) = 2,\ B''(1) = 4$ and $B'''(0) = 0$, the IS in (8) produces a CO four. For instance, choose $B(t) = 1 - 2(t-1) - 2(t-1)^2$ and we have a new IS (OA4) as:

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)}$$

$$w_{n+1} = y_n - \left[1 - 2\left(\frac{\Gamma(y_n)}{\Gamma(w_n)} - 1\right) - 2\left(\frac{\Gamma(y_n)}{\Gamma(w_n)} - 1\right)^2\right]\frac{\Gamma(y_n)}{\varphi(w_n)}. \quad (16)$$

Theorem 2: Assume the conditions imposed on $\Gamma(w)$ holds as in Theorem 1, then the sequence $\{w_n\}_{n\geq 0}, (w_n \in I)$ of iterative outputs produced by the family of IS in (11) converges to θ with CO seven when $B(t)$, $H(t)$ and $G(v)$ are imposed with the conditions $B(1) = G(0) = H(0) = 1$, $B'(1) = -2$, $B''(1) = -4$ and $B'''(0) = 0$, $G''(0) < \infty$, $H'(0) = 0$, $H''(0) = 2, H'''(0) = 24$, $H^{(iv)}(1) < \infty$.

Proof: Here, the procedure used in the proof of Theorem 1 is followed. For the sake of the program, let Γz be $\Gamma(z)$. The program for the convergence test is put forward next:

$\Gamma w = (d + c_2 * d^2 + c_3 * d^3 + c_4 * d^4 + c_5 * d^5 + c_6 * d^6 + c_7 * d^7 + c_8 * d^8);$
$A = d + \delta(\Gamma w)^3;$
$\Gamma A = (A + c_2 * A^2 + c_3 * A^3 + c_4 * A^4 + c_5 * A^5 + c_6 * A^6 + c_7 * A^7 + c_8 * A^8);$
$M = \text{Series}\left[\left(\frac{\Gamma A - \Gamma w}{A - d}\right), \{d, 0,7\}\right] //\text{Simplify};$
$y = \text{Series}\left[e - \left(\frac{\Gamma w}{M}\right), \{d, 0,7\}\right] //\text{Simplify};$
$\Gamma y = (y + c_2 * d^2 + c_3 * d^3 + c_4 * d^4);$
$t = \text{Series}\left[\left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0,7\}\right] //\text{Simplify};$
$B = \text{Series}[1 - 2 * (t - 1) - 2 * (t - 1)^2, \{d, 0,7\}]//\text{Simplify};$
$z = \text{Series}\left[y - B * \left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0,4\}\right] //\text{Simplify};$
$\Gamma z = \text{Series}[(z + c_2 * z^2 + c_3 * z^3 + c_4 * z^4), \{d, 0,4\}]//\text{Simplify};$
$v = \text{Series}\left[\left(\frac{\Gamma z}{\Gamma y}\right), \{d, 0,7\}\right] //\text{Simplify};$
$G = \text{Series}[1 + v, \{d, 0,7\}]//\text{Simplify};$
$H = \text{Series}[1 + t^2 + 4 * t^3, \{d, 0,7\}]//\text{Simplify};$
$w = \text{Series}\left[z - B * \left(\frac{\Gamma z}{M}\right) * H * G, \{d, 0,7\}\right] //\text{Simplify}$
The program output is:

$$w_{n+1} = \theta + 2c_2^2(-7c_2^2 + c_3)d_n^7 + O(e_n^8). \quad (17)$$

Again, by Definition 1, the IS in (9) has the equation in (17) as error equation. As a result of this, its CO is seven. This verified the claim in Theorem 2.

Remark 2

For some RVWF $B(t)$, $G(v)$ and $H(t)$ jointly satisfying the conditions in Theorem 2, will produce a CO seven IS. Suppose $B(t) = 1 - 2(t-1) - 2(t-1)^2$, $G(v) = 1 + v$ and $H(t) = 1 + t^2 + 4t^3$, $\alpha \in R$, we obtain a new order seven IS(OA7) given as:

$$\begin{aligned} y_n &= w_n - \frac{\Gamma(w_n)}{\varphi(w_n)}; \\ z_n &= y_n - B(t)\frac{\Gamma(y_n)}{\varphi(w_n)}; \\ w_{n+1} &= z_n - B(t)\frac{\Gamma(z_n)}{\varphi(w_n)}\left[1 + \frac{\Gamma(z_n)}{\Gamma(y_n)}\right]\left[1 + \left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^2 + 4\left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^3\right]. \end{aligned} \tag{18}$$

Theorem 3: Consider the conditions imposed on $\Gamma(w)$ in Theorem 1 holds, then for any w_0 close to θ, a sequence $\{w_n\}_{n\geq 0}$, $(w_n \in I)$ of iterative outputs produced by the family of IS in (13) converges to θ with CO eight when $B(t)$, $G(v)$, $H(t)$ and $Q(s)$ is imposed with the conditions that $B(1) = G(0) =, G'(0) = H(0) = Q(0) = 1$, $B'(1) = -2$, $B''(1) = -4$, $B'''(0) = 0$, $H'(0) = 0, Q'(0) = 2$, $H''(0) = -20$, $Q''(0) < \infty$.

Proof: The convergence error is obtained using the program below:

$\Gamma w = (d + c_2 * d^2 + c_3 * d^3 + c_4 * d^4 + c_5 * d^5 + c_6 * d^6 + c_7 * d^7 + c_8 * d^8);$
$A = d + \delta(\Gamma w)^3;$
$\Gamma A = (A + c_2 * A^2 + c_3 * A^3 + c_4 * A^4 + c_5 * A^5 + c_6 * A^6 + c_7 * A^7 + c_8 * A^8);$
$M = \text{Series}\left[\left(\frac{\Gamma A - \Gamma w}{A - d}\right), \{d, 0,8\}\right] //\text{Simplify};$
$y = \text{Series}\left[d - \left(\frac{\Gamma w}{M}\right), \{d, 0,8\}\right] //\text{Simplify};$
$\Gamma y = (y + c_2 * d^2 + c_3 * d^3 + c_4 * d^4);$
$t = \text{Series}\left[\left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0,8\}\right] //\text{Simplify};$
$B = \text{Series}[1 - 2 * (t-1) - 2 * (t-1)^2, \{d, 0,8\}] //\text{Simplify};$
$z = \text{Series}\left[y - B * \left(\frac{\Gamma y}{\Gamma w}\right), \{d, 0,8\}\right] //\text{Simplify};$
$\Gamma z = \text{Series}[(z + c_2 * z^2 + c_3 * z^3 + c_4 * z^4), \{d, 0,8\}] //\text{Simplify};$
$v = \text{Series}\left[\left(\frac{\Gamma z}{\Gamma y}\right), \{d, 0,8\}\right] //\text{Simplify};$
$G = \text{Series}[1 + v, \{d, 0,8\}] //\text{Simplify};$
$H = \text{Series}[1 + t^2 - 10 * t^3, \{d, 0,8\}] //\text{Simplify};$
$s = \text{Series}\left[\left(\frac{\Gamma z}{\Gamma w}\right), \{d, 0,8\}\right] //\text{Simplify};$
$Q = \text{Series}[1 + 2 * s + Q''(0) * s^2, \{d, 0,8\}] //\text{Simplify};$

$$\mathrm{w} = \mathrm{Series}\left[\mathrm{z} - \mathrm{B} * \left(\frac{\Gamma \mathrm{z}}{\mathrm{M}}\right) * \mathrm{H} * \mathrm{G} * \mathrm{Q}, \{\mathrm{d}, 0{,}8\}\right] //\mathrm{Simplify}$$

The program output is:

$$w_{n+1} = \theta + \frac{1}{24}\left[c_2(7c_2^2 - c_3)(\delta c_2^2 + 64c_2^4 - 18c_2^2c_3 + c_3^2 + c_2c_4)\right]d_n^8 + O(e_n^9). \quad (19)$$

The equation in (19) is the IS error equation. From the error equation given in Definition 1, it can be deduced from the expression in (19), that the family of IS in (11) has a general CO of eight.

Remark 3

For RVWF $B(t), G(t),\ G(t)$ and $G(s)$ satisfying the conditions in Theorem 3 will produces a CO eight. Suppose $B(t) = 1 - 2(t-1) - 2(t-1)^2,\ G(v) = 1 + v$ and $H(t) = 1 + t^2 - 10t^3,\ Q(s) = 1 + 2s + \tau s^2, \tau \in R$ we obtain the new IS OA8 given as:

$$y_n = w_n - \frac{\Gamma(w_n)}{\varphi(w_n)};$$

$$z_n = y_n - B(t)\frac{\Gamma(y_n)}{\varphi(w_n)};$$

$$w_{n+1} = z_n - B(t)\frac{\Gamma(z_n)}{\varphi(w_n)}\begin{bmatrix} 1 + \left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^2 \\ -10\left(\frac{\Gamma(z_n)}{\Gamma(w_n)}\right)^3 \end{bmatrix}\left[1 + \frac{\Gamma(z_n)}{\Gamma(y_n)}\right]\begin{bmatrix} 1 + 2\left(\frac{\Gamma(z_n)}{\Gamma(w_n)}\right) \\ +\tau\left(\frac{\Gamma(y_n)}{\Gamma(w_n)}\right)^2 \end{bmatrix}. \quad (20)$$

3.0 RESULTS

The implementation and applicability of the developed IS (OA4, OA7, and OA8) are illustrated in this section. Computation programs for OA4, OA7, OA8a (using $\tau = 0$ in Eq. (20)) and OA8b (using $\tau = 12$ in Eq. (20)) were written in MAPLE 9 environment and used to locate the solution of some NL equations. Their computation outputs were juxtaposed with that of some existing IS (with same CO) outputs (see Tables 1 and 2). The compared IS includes the CO four Kung-Traub (1974) (KT4), CO seven IS presented in Panday and Jaiswal (2017) (PJ7), CO eight IS in Soleymani (2011) (SL8) and the CO eight IS of Yasmin et al.,(2016) (YZS8).

The metric used for contrast includes, the IS required number of iteration rounds to attain convergence (IT), residual function of last iteration $|f(w_{n+1})|$ and computational order of convergence φ_{coc} defined in Jay(2001) (see Definition 3.3).

The precision digits and error bound used in the programing are 2000 significant figures and $\varepsilon = 10^{-1000}$ respectively. The NL equations used for the test are:

$$\Gamma_1(w) = \sqrt{w^2 + 2w + 5} - 2\sin w + w^2 + 3\,, \quad \theta = 2.331967\ldots$$

$$\Gamma_2(w) = e^{(w^2-w)} - \cos(1-w^2) + w^3 + 1\,, \qquad \theta = -1$$

Table 1. Iterative schemes results comparison for $\Gamma_1(w)$ and $\Gamma_2(w)$.

$\Gamma_n(w)$	IS	w_0	IT	$\lvert\Gamma(w_{n+1})\rvert$	φ_{coc}
$\Gamma_1(w)$	$KT4_{\delta=.001}$	1.5	5	6.8894e-1193	4
	$OA4_{\delta=.001}$		5	5.3424e-1235	4
	PJ7		4	1.1103e-0431	7
	$OA7_{\delta=.001}$		3	3.3332e-0377	7
	SL7		4	8.4289e-1842	7
	YZS8		Failed	-	-
	$OA8a_{\delta=.001}$		3	7.5523e-0493	8
	$OA8b_{\delta=.001}$		3	7.4711e-0493	8
$\Gamma_2(w)$	$KT4_{\delta=.0001}$	0.4	7	9.1016e-0857	4
	$OA4_{\delta=.0001}$		6	2.0764e-1511	4
	PJ7		Failed	-	-
	$OA7_{\delta=.0001}$		5	3.2173e-1488	7
	SL7		5	7.5632e-0765	7
	YZS8		Failed	-	-
	$OA8a_{\delta=.0001}$		4	1.6975e-0465	8
	$OA8b_{\delta=.0001}$		4	2.8023e-0453	8

Next, the IS were applied to solve some real life models.

Model 1: Environmental engineering model (Chapra and Canale, 2020)

Consider the equation describing the downstream oxygen level C(mg/L) when sewage is discharged in a river given as

$$\mathrm{C} = 10 - 20\,(\exp(-.15\mathrm{w}) - \exp(-.5\mathrm{w}))$$

where w (in kilometers) represents distance downstream that is desired to be known. A problem may arise to determine the distance downstream (w) when the oxygen level falls to $5\mathrm{mg/L}$. In this case, the above model becomes

$$\Gamma_3(\mathrm{w}) = 20\,(\exp(-.15\mathrm{w}) - \exp(-.5\mathrm{w})) - 5,$$

and its real solution is $\mathrm{w} = 0.9762 \ldots$

Model 2: Aerospace engineering model (Kallrath, 2002).

Consider the Kepler's model in astronomy given as

$$\mathrm{M} = \mathrm{w} - \mathrm{e}\sin\mathrm{w}, \quad \mathrm{M} \in [0,1],$$

where M and e are mean anomaly and e eccentricity respectively and play a significant role in the Kepler's model. The eccentric anomaly w often used to determine the position of a moving point in a Keplerian orbit. A case may require to determine the eccentric anomaly w when $\mathrm{M} = 0.6$ and $\mathrm{e} = 0.9$. In the case, the Kepler's model become:

$$\Gamma_4(\mathrm{w}) = \mathrm{w} - 0.9\sin\mathrm{w} - 0.6.$$

The actual solution of the above equation is $\mathrm{w} = 1.4975 \ldots$

The computation results when the developed IS and compared schemes are presented in Table 2.

Table 2. Iterative schemes results comparison for $\Gamma_3(\mathrm{w})$ and $\Gamma_4(\mathrm{w})$.

$\Gamma_n(\mathrm{w})$	IS	w_0	IT	$\lvert\Gamma(\mathrm{w}_{n+1})\rvert$	φ_{coc}
	$\mathrm{KT4}_{\delta=-.001}$		5	1.5185e-1013	4
	$\mathrm{OA4}_{\delta=-.001}$		5	1.3414e-0766	4
	PJ7		4	5.0996e-0750	7

$\Gamma_3(w)$	$OA7_{\delta=-.001}$	1.2	4	3.4046e-1825	7
	SL7		4	4.9214e-0840	7
	YZS8		3	4.7383e-0246	8
	$OA8a_{\delta=-.001}$		3	3.2424e-0326	8
	$OA8b_{\delta=-.001}$		3	4.7633e-0329	8
$\Gamma_4(w)$	$KT4_{\delta=1}$	0.4	6	4.0530e-1579	4
	$OA4_{\delta=1}$		5	3.6579e-0673	4
	PJ7		4	2.3479e-1579	7
	$OA7_{\delta=1}$		4	1.9707e-0933	7
	SL7		4	1.0886e-0727	7
	YZS8		3	2.0746e-0370	8
	$OA8a_{\delta=1}$		4	2.3025e-1117	8
	$OA8b_{\delta=1}$		4	2.6170e-1273	8

3.1 Results Discussion

The symbol IS_δ used in Table 4.1-4.2, denotes an Iterative Structure when the value of the parameter δ as indicated is used. The computation outputs are presented in tables in the form $De-E$ to implies $D \times 10^E$ where $D, E \in R$. From Table 1 and Table 2, it can be observed that our contributed IS (OA4, OA7,OA8a and OA8b) at varied values of parameters, solved all the tested examples with competitive level of precision. In fact, in Table 1 where the compared IS PJ7 and YZS8 failed to provide solutions, all the IS introduced in this work, obtained the solutions with high convergence. The implication is that our IS can be used where existing IS failed to solve NL models. Finally, on all the problem tested, the φ_{coc} produced by the various developed IS (see the last column of Table 1 and Table 2) coincided with their respective convergence order values obtained from the theoretical analysis in Section 3.

4.0 CONCLUSION

We have applied the divided difference, weight function and composition techniques to modifying the two step IS in Alfio et al.,(2000). Consequently, a new and optimal families of IS with CO four, seven and eight for determining the solution of real life models are put forward. The convergence of the developed IS have been proven via written programs in MATHEMATICA software. The implementation of the developed IS on some real life phenomena, shows that they are efficient and can be regarded as veritable tools for quantitative assessment of sustainability problems modeled into nonlinear equations.

For further studies, the complex dynamical and chaotic behavior of the developed schemes is a green area for exploit.

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